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On the convergence rate of 2-dimensional low discrepancy sequences

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1 Overview

We will consider a sequence $\{x_N\}_{N=1}^{\infty}$ which is uniformly distributed on $[0,1]^d$, that is, for any interval $J \in [0,1]^d$,

$$\lim_{N\to\infty} \frac{\#\{x_i \in J \mid i \le N\}}{N} = |J|.$$

Here, $|J|$ is the Lebesgue measure of J. The discrepancy of $\{x_N\}_{N=1}^{\infty}$ is defined by

$$D(N) := \sup_J \left| \frac{\#\{x_i \in J \mid i \le N\}}{N} - |J| \right|,$$

here $\sup_J$ is taken over all intervals $\prod_{i=1}^{d}[0,l_i)$ $(0 < l_i \le 1)$. It is conjectured that for any sequence $\{x_N\}_{N=1}^{\infty}$

$$D(N) \ge O\left(\frac{(\log N)^d}{N}\right).$$

We call $\{x_N\}_{N=1}^{\infty}$ a *low discrepancy* sequence if it satisfies

$$D(N) = O\left(\frac{(\log N)^d}{N}\right).$$

For a low discrepancy sequence $\{x_N\}_{N=1}^{\infty}$, $D(N)/\left(\frac{(\log N)^d}{N}\right)$ is bounded. In quasi Monte Carlo methods, we need the sequence with small bound. Thus we are interested in this rate for concrete sequences.

In the following sections, we consider the case $d=2$, and introduce two low discrepancy sequences and make numerical experimentations of their bounds.

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2 Halton Sequences (multi–dimensional van der Corput sequences)

In this section, we will review the construction of Halton sequences, one of the most famous low discrepancy sequences. Let p be a positive integer and $n_{k,N}$ be k-th number of p-adic expansion of N: $N = \sum_{k=0}^{\infty} n_{k,N}p^k$. Using this $n_{k,N}$, we construct x_N by an equation $x_N = \sum_{k=0}^{\infty} n_{k,N}p^{-k-1}$. Then $\{x_N\}_{N=1}^{\infty}$ is a low discrepancy sequence called van der Corput sequence. For d dimensional cases, take different primes $p_1, p_2, \cdots, p_d$, and expand N to p_j-adic for every $j = 1, 2, \ldots d$. For each expansion, we set x_N^j as above:

$$x_N^j = \sum_{k=0}^{\infty} n_{k,N}^j p_j^{-k-1}.$$

Then the sequence $\left\{(x_N^1, x_N^2, \cdots, x_N^d)\right\}_{N=1}^{\infty}$ is proved to be a d-dimensional low discrepancy sequence.

Let us consider an example. For 2-dimensional case, choose $p_1 = 2$ and $p_2 = 3$. Then

$$\begin{array}{lll}
 & p_1 = 2,\ \text{binary expansion} & p = 3,\ \text{ternary expansion} \\
1 \to & 01 \to x_1 = 0.1_{(2)} = \frac{1}{2} & 01 \to x_1 = 0.1_{(3)} = \frac{1}{3} \\
2 \to & 10 \to x_1 = 0.01_{(2)} = \frac{1}{4} & 02 \to x_1 = 0.2_{(3)} = \frac{2}{3} \\
3 \to & 11 \to x_1 = 0.11_{(2)} = \frac{3}{4} & 10 \to x_1 = 0.01_{(3)} = \frac{1}{9} \\
4 \to & 100 \to x_1 = 0.001_{(2)} = \frac{1}{8} & 11 \to x_1 = 0.11_{(3)} = \frac{4}{9} \\
 & \vdots &
\end{array}$$

Then, a Halton sequence is:

$$\left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{4}, \frac{2}{3}\right), \left(\frac{3}{4}, \frac{1}{9}\right), \left(\frac{1}{8}, \frac{4}{9}\right), \cdots$$

3 Mori's Sequences

In this section, we introduce Mori's sequence, which is two dimensional sequence constructed by the inverse image of symbolic dynamics. To express the dynamics, we prepare some notations. Let p be an integer, and $\mathcal{A} = \{0, 1, 2 \ldots, p-1\}$. $\mathcal{A}$ is the set of p-adic digits. We treat one sided infinite sequence $x = x_1 x_2 \cdots x_k \cdots$, $(x_i \in \mathcal{A})$, and identify it with $\sum_{n=1}^{\infty} x_n p^{-n} \in [0, 1]$. We denote by $(x)_i$ the i-th symbol of a sequence $x = x_1 x_2 \cdots x_k \cdots$. For two sequences x and y, we define the sum $z = x + y$ by digitwise sum: $(z)_i = (x)_i + (y)_i \pmod p$, $i = 1, 2, \ldots$. We express the shift operator by θ, that is, $\theta x_1 x_2 \cdots = x_2 x_3 \cdots$.

For a finite word $w = a_1 a_2 \cdots a_k$ and an infinite word $x = x_1 x_2 \cdots$, we define $wx = a_1 a_2 \cdots a_k x_1 x_2 \cdots$. Similarly

$$\begin{pmatrix} a_1 a_2 \cdots a_k \\ a_1' a_2' \cdots a_k' \end{pmatrix} (x_1 x_2 \cdots, x_1', x_2' \cdots) = \begin{pmatrix} a_1 a_2 \cdots a_k, x_1 x_2 \cdots \\ a_1' a_2' \cdots a_k', x_1 x_2 \cdots \end{pmatrix}$$

We need infinite sequences $S_0, S_1, \cdots, S_{p-1}$ of $\mathcal{A}$ with the following properties:

1. For any positive integer k and any finite word $w = a_1 a_2 \cdots a_k$, there exists a unique $S_{i_1}, S_{i_2}, \cdots, S_{i_k}$, such that first k symbols of $S_{i_1} + \theta S_{i_2} + \cdots + \theta^{k-1} S_{i_k}$ coincide with w.
2. For each $i, j \in \mathcal{A}$, $S_i + S_j = S_{i+j} \pmod{p}$ (digitwise).

We can prove the existence of such sequences. For $p = 2$, one of the examples of these sequences S_0 and S_1 are

$$S_0 = 0 \cdots,$$
$$S_1 = 101000100000010 \cdots,$$

that is,

$$(S_1)_i = \begin{cases} 1 & \text{if } k = 2^n - 1 \ (n = 1, 2, \ldots), \\ 0 & \text{otherwise.} \end{cases}$$

Now we can define transformations F_n $(n = 1, 2, \ldots)$ on $[0,1]^2$ as follows:

$$F_n \begin{pmatrix} x \\ y \end{pmatrix} = \theta^n \begin{pmatrix} i_1^x, i_2^x, \cdots, i_k^x, \cdots \\ i_1^y, i_2^y, \cdots, i_k^y, \cdots \end{pmatrix} + \begin{pmatrix} S_{i_n^y} + \theta S_{i_{n-1}^y} + \cdots + \theta^n S_{i_1^y} \\ S_{i_n^x} + \theta S_{i_{n-1}^x} + \cdots + \theta^n S_{i_1^x} \end{pmatrix},$$

here $\begin{pmatrix} i_1^x, i_2^x, \cdots, i_k^x, \cdots \\ i_1^y, i_2^y, \cdots, i_k^y, \cdots \end{pmatrix}$ is the p-adic expansion of $\begin{pmatrix} x \\ y \end{pmatrix} \in [0,1]^2$. Then the inverse image of F_n is expressed

$$F_n^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varepsilon_1^x, \varepsilon_2^x, \cdots, \varepsilon_n^x \\ \varepsilon_1^y, \varepsilon_2^y, \cdots, \varepsilon_n^y \end{pmatrix} \left\{ \begin{pmatrix} i_1^x, i_2^x, \cdots, i_k^x, \cdots \\ i_1^y, i_2^y, \cdots, i_k^y, \cdots \end{pmatrix} + \begin{pmatrix} S_{\varepsilon_n^y} + \theta S_{\varepsilon_{n-1}^y} + \cdots + \theta^n S_{\varepsilon_1^y} \\ S_{\varepsilon_n^x} + \theta S_{\varepsilon_{n-1}^x} + \cdots + \theta^n S_{\varepsilon_1^x} \end{pmatrix} \right\}$$

for $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i_1^x, i_2^x, \cdots, i_k^x, \cdots \\ i_1^y, i_2^y, \cdots, i_k^y, \cdots \end{pmatrix}$. Take any initial point $\begin{pmatrix} x \\ y \end{pmatrix}$ and fix it. Because we can choose $\begin{pmatrix} \varepsilon_1^x, \varepsilon_2^x, \cdots, \varepsilon_n^x \\ \varepsilon_1^y, \varepsilon_2^y, \cdots, \varepsilon_n^y \end{pmatrix}$ arbitrary, the number of the inverse images under F_n equals p^{2n}.

To align these inverse images, we introduce an order. First, we define an order in $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathcal{A}^2$ as follows:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} < \begin{pmatrix} 1 \\ 0 \end{pmatrix} < \begin{pmatrix} 2 \\ 0 \end{pmatrix} < \cdots < \begin{pmatrix} p-1 \\ 0 \end{pmatrix} < \begin{pmatrix} 0 \\ 1 \end{pmatrix} < \begin{pmatrix} 1 \\ 1 \end{pmatrix} < \cdots < \begin{pmatrix} p-1 \\ 1 \end{pmatrix} < \begin{pmatrix} 0 \\ 2 \end{pmatrix} < \cdots \begin{pmatrix} p-1 \\ p-1 \end{pmatrix}. \tag{1}$$

Using this order, we define an order in $\mathcal{A}^{2k}$. We determine

$$\begin{pmatrix} a_1, a_2, \cdots, a_k \\ a_1', a_2', \cdots, a_k' \end{pmatrix} < \begin{pmatrix} b_1, b_2, \cdots, b_k \\ b_1', b_2', \cdots, b_k' \end{pmatrix},$$

if there exists $l < k$ such that

$$\begin{pmatrix} a_{l+1}, \cdots, a_k \\ a_{l+1}', \cdots, a_k' \end{pmatrix} = \begin{pmatrix} b_{l+1}, \cdots, b_k \\ b_{l+1}', \cdots, b_k' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_l \\ a_l' \end{pmatrix} < \begin{pmatrix} b_l \\ b_l' \end{pmatrix}$$

in the order (1). Moreover, for two words with different length $k < l$, we define $\begin{pmatrix} a_1, \cdots, a_k \\ a'_1, \cdots, a'_k \end{pmatrix} < \begin{pmatrix} b_1, \cdots, b_l \\ b'_1, \cdots, b'_l \end{pmatrix}$. Along this order, we align the inverse image of F_k $(k = 1, 2, \ldots)$. Namely,

$$
\begin{array}{l}
\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} (x,y), \cdots, \begin{pmatrix} p-1 \\ p-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \\
\begin{pmatrix} 00 \\ 00 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 10 \\ 00 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 20 \\ 00 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \cdots, \begin{pmatrix} (p-1)0 \\ (p-1)0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \\
\begin{pmatrix} 01 \\ 00 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 11 \\ 00 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 21 \\ 00 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \cdots, \begin{pmatrix} (p-1)1 \\ (p-1)0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \\
\qquad \cdots \\
\begin{pmatrix} 0(p-1) \\ 0(p-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 1(p-1) \\ 0(p-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 2(p-1) \\ 0(p-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \cdots, \begin{pmatrix} (p-1)(p-1) \\ (p-1)(p-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \\
\qquad \cdots.
\end{array}
$$

Then this sequence is proved to be a low discrepancy sequence. ([5])

4 Numerical Results

We will compare the discrepancies and the rates of convergence

$$C = \lim_{N \to \infty} D(N) / \left(\frac{(\log N)^2}{N} \right), \tag{2}$$

for two low discrepancy sequences which we studied in the previous sections. All calculations were carried out in Mathematica 4.2 on Windows-XP and Mathematica 5.0 on Macintosh OS X.

We make the first 1,000,000 number of each sequences, under the following conditions: For Halton sequences,

case (1). $p_1 = 2$, $p_2 = 3$,
case (2). $p_1 = 17$, $p_2 = 19$,
case (3). $p_1 = 71$, $p_2 = 73$.

These are the examples, (1) small primes, (2) medium primes, and (3) large primes.

For Mori's sequences, $p = 2$ and

case (1). initial point $(0, 0)$,
case (2). initial point $(0, 4, 0, 4)$,
case (3). initial point $(\frac{\pi}{10}, \frac{\sqrt{2}}{2})$.

These are the examples of initial points, (1) the simplest case, (2) a rational case, and (3) an irrational case.

We calculate the discrepancy of each sequence of the visit to an interval $[0, x] \times [0, y]$ in the following cases:

case (A). $[0, 0.4] \times [0, 0.4]$,
case (B). $[0, \frac{\pi}{10}] \times [0, \frac{1}{\sqrt{2}}]$,
case (C). $[0, 0.1] \times [0, 0.8]$,
case (D). $[0, 0.5] \times [0, 0.5]$.

Here, we choose two square cases ((A) and (D)), an irrational rectangle (B), and a rectangle for which the length of edges are quite different (C).

The numerical results are shown in Table 1 and Table 2. In these tables, the upper row corresponds to discrepancies $D(N)$ and the lower row corresponds to rates $D(N)/((\log N)^2/N)$ at $N = 1000000$ for each columns.

Table 1: Results of Halton sequences

	case (A)	case (B)	case (C)	case (D)
case (1)	1.0×10^{-5} 0.05	1.0×10^{-5} 0.05	6.0×10^{-6} 0.05	6.0×10^{-6} 0.05
case (2)	2.0×10^{-5} 0.1	2.0×10^{-5} 0.1	1.0×10^{-5} 0.05	
case (3)	1.2×10^{-4} 0.6	1.4×10^{-4} 0.7	3.5×10^{-5} 0.18	

Table 2: Results of Mori's sequences

	case (A)	case (B)	case (C)	case (D)
case (1)	2.4×10^{-4} 1.6	2.7×10^{-4} 1.3	4.0×10^{-5} 1.8	8.0×10^{-7} 0.004
case (2)		8.0×10^{-5} 0.5		
case (3)	2.3×10^{-4} 1.2	2.5×10^{-4} 1.3	4.0×10^{-4} 1.9	

In the case Mori's case D, the rate is very small. This is because Mori's sequences based on binary expansion, and we consider the visits to the square $[0, \frac{1}{2}]^2$.

The distributions in $[0, 1]^2$ are shown from Fig. 1 to Fig. 4. The first 1000 points in each sequence are plotted in the figures. It seems that Halton sequences distribute entirely to $[0, 1]^2$ in small primes and are getting worse in large primes as shown in Fig. 2, and that Mori's sequences may not depend on initial points.

Discrepancies $D(N)$ and the ratios $D(N)/((\log N)^2/N)$ are shown in Fig. 5 to Fig. 20 These figures are the first 100000 discrepancies, the last 100000 discrepancies, the first 100000 ratios and the last 100000 ratios of million data for each case. Because the speed of convergence to these ratios is very slow, it seems that the ratios are still decreasing after million data in all cases.

Fig. 5 to Fig. 9 are the Discrepancies $D(N)$ and the ratios $D(N)/((\log N)^2/N)$ of the visits to $[0, 0.4]^2$ for Halton sequences with $p_1 = 2$, $p_2 = 3$. Fig. 9 to Fig. 12 are the discrepancies and the ratios of the visits to $[0, 0.1] \times [0, 0.8]$ for Halton sequences with $p_1 = 71$, $p_2 = 73$. Fig. 13 to Fig. 16 are the discrepancies and the ratios of the visits to $[0, 0.4]^2$ for Mori's sequence with the initial point $(0.4, 0.4)$ Fig. 17 to Fig. 20 are the discrepancies and the ratios of the visits to $[0, 0.5]^2$ for Mori's sequence with the initial point $(0.4, 0.4)$. In these figures, four lines appear. This means for the interval $[0, \frac{1}{2}]^2$, discrepancy oscillates with period $4 = 2^2$.

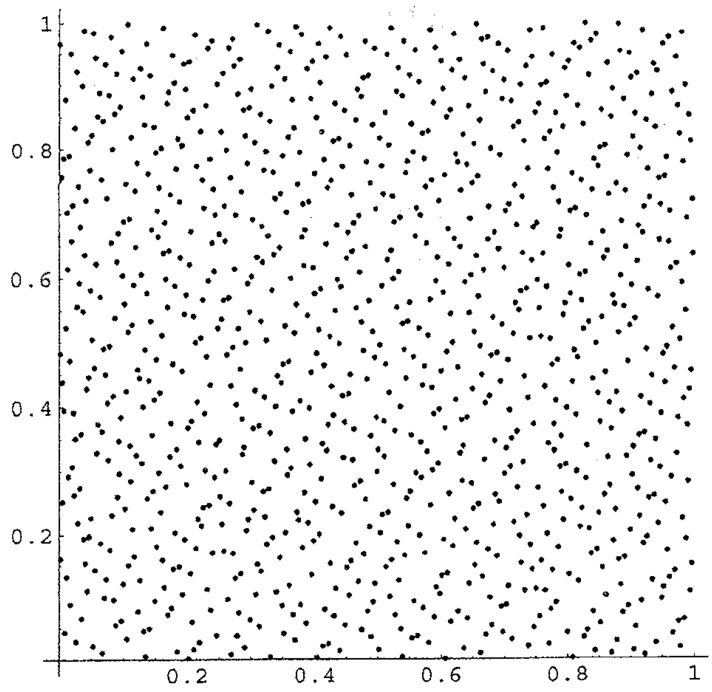

Figure 1: 1000 plots of Halton sequence, $p_1 = 2$, $p_2 = 3$

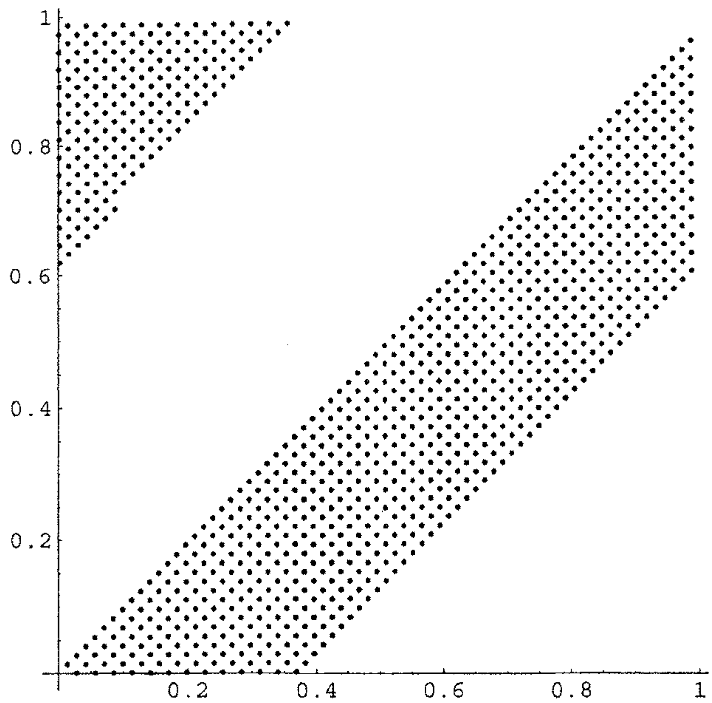

Figure 2: 1000 plots of Halton $p_1 = 71$, $p_2 = 73$

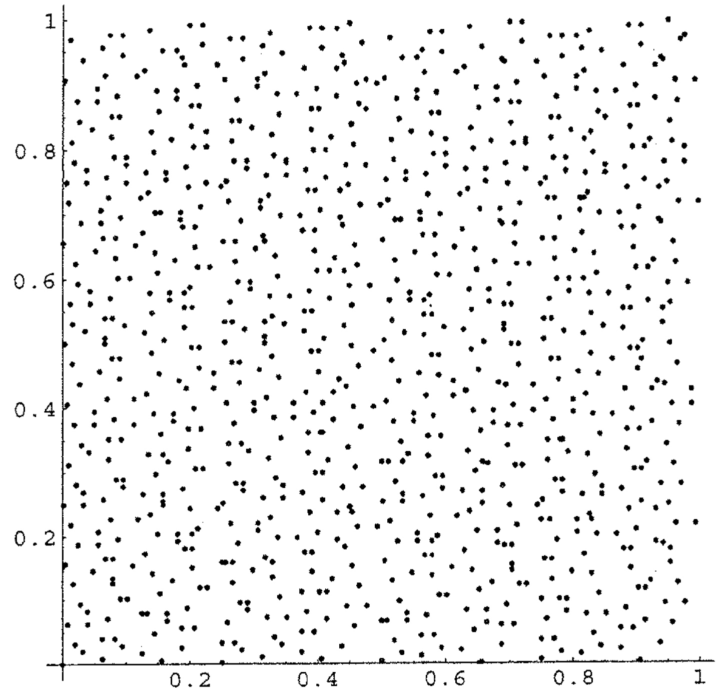

Figure 3: 1000 plots of Mori, initial point $(0, 0)$

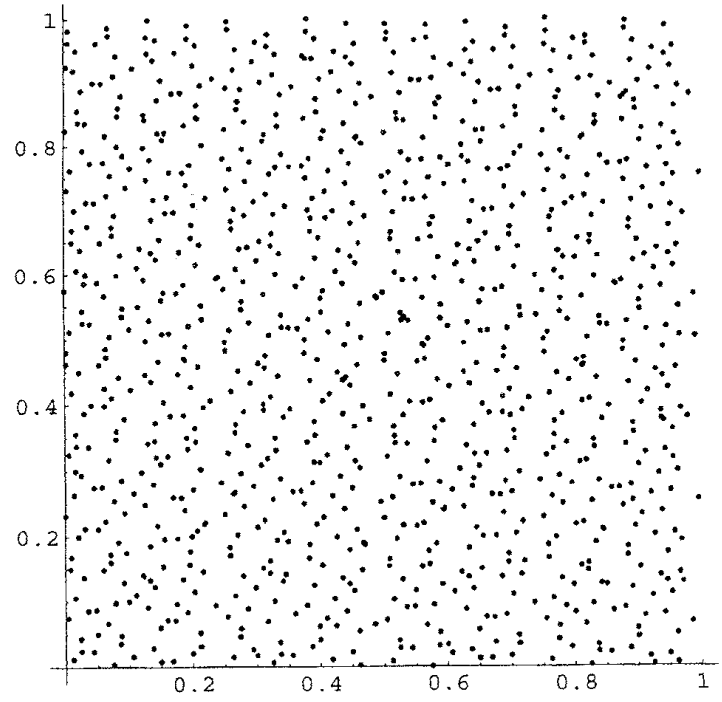

Figure 4: 1000 plots of Mori, initial point $(0.4, 0.4)$

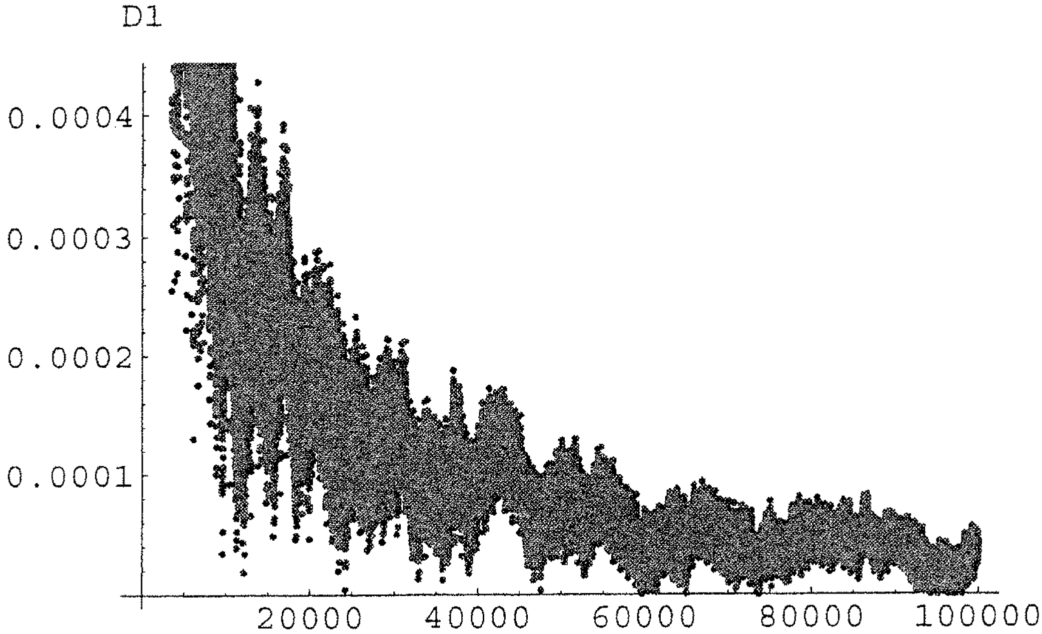

Figure 5: Halton $p_1 = 2$, $p_2 = 3$, discrepancy $D(N)$ of the first 100000 of Million data

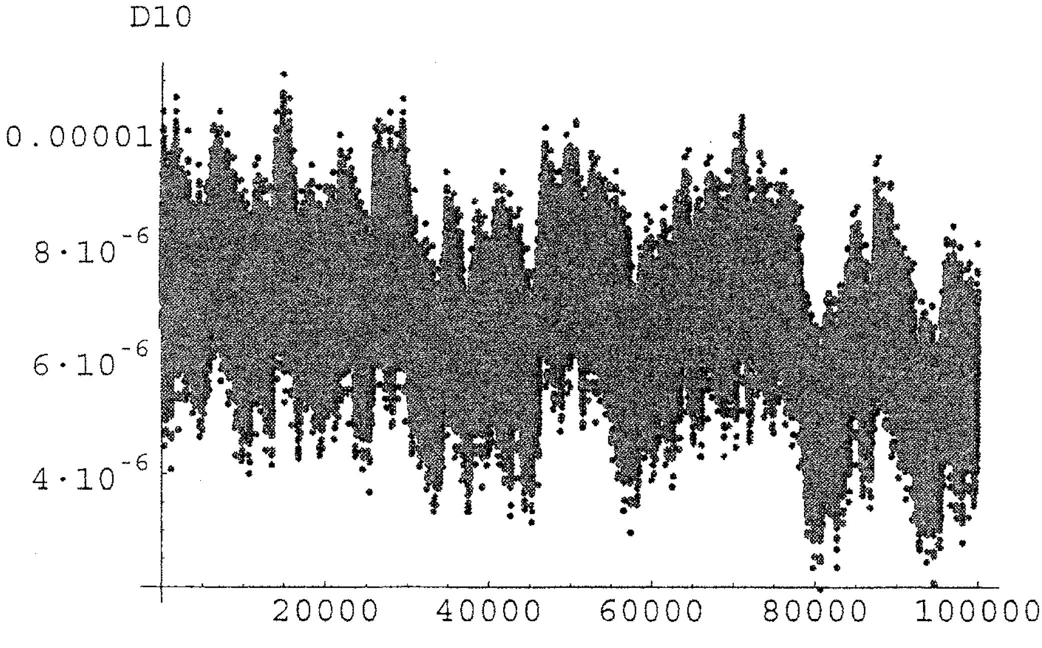

Figure 6: Halton $p_1 = 2$, $p_2 = 3$, discrepancy of the last 100000 of Million data

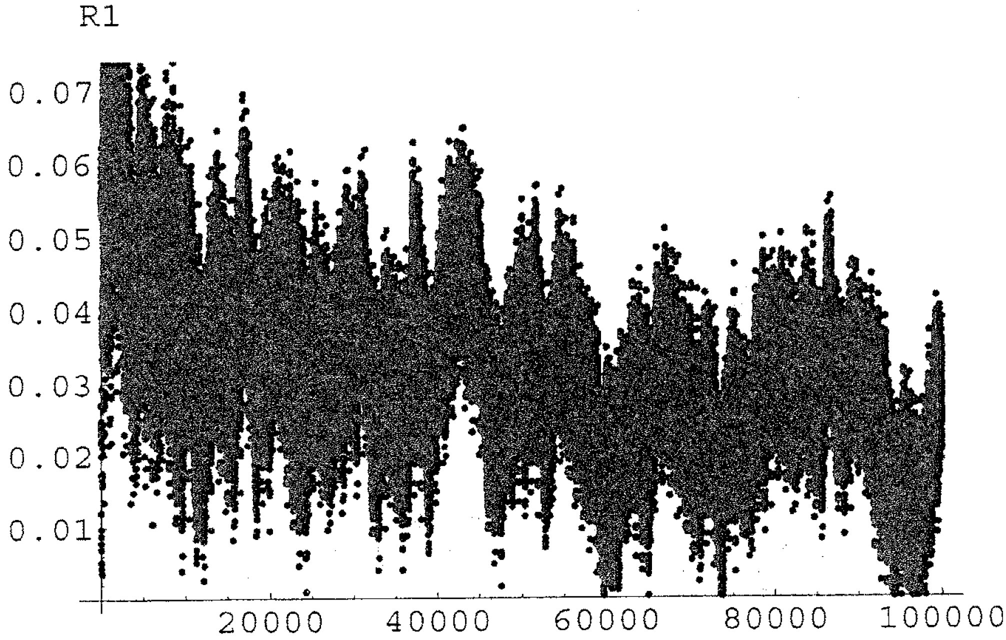

Figure 7: Halton $p_1 = 2$, $p_2 = 3$ ratio of the first 100000

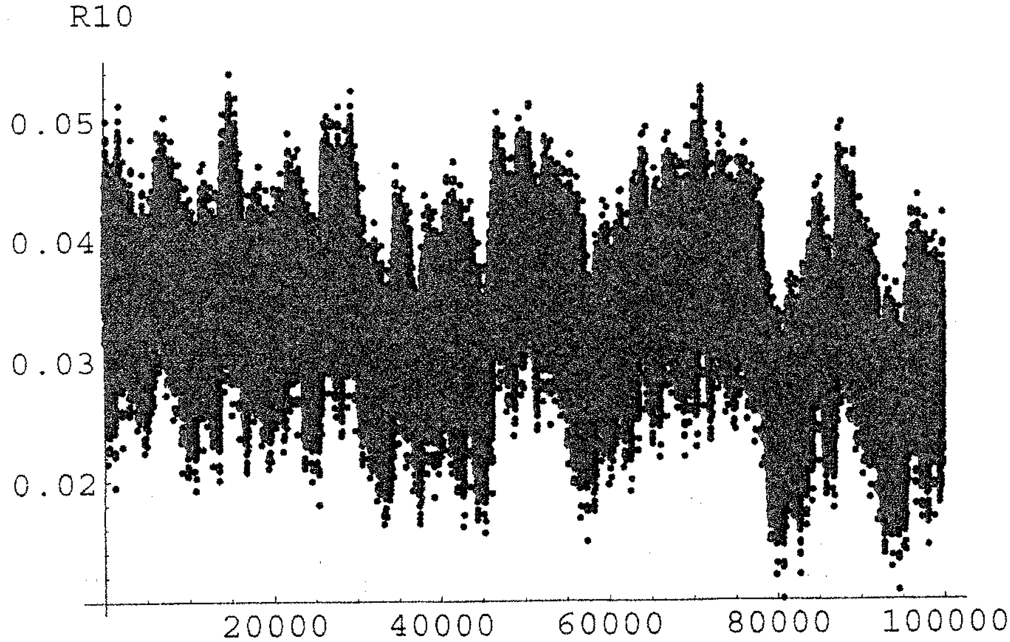

Figure 8: Halton $p_1 = 2$, $p_2 = 3$ ratio of the last 100000

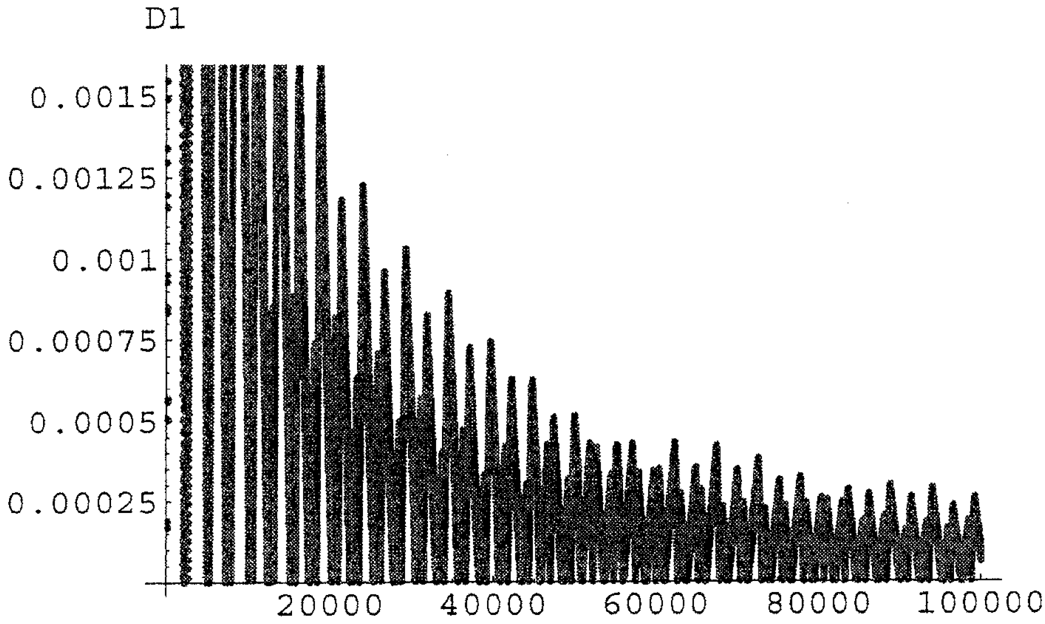

Figure 9: Halton $p_1 = 71$, $p_2 = 73$, discrepancy: the first 100000 of Million data

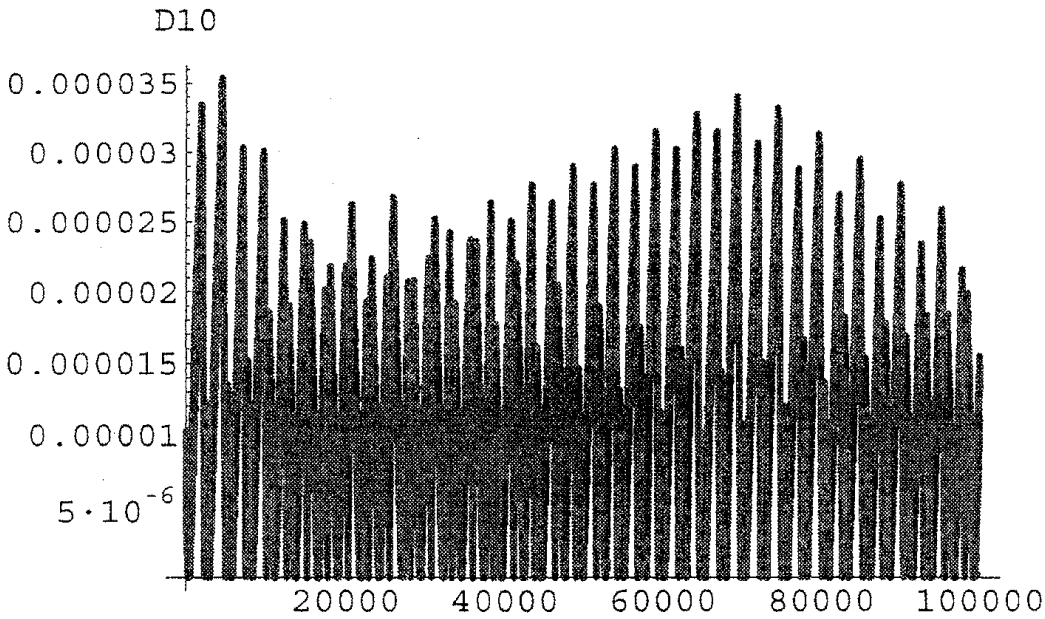

Figure 10: Halton $p_1 = 71$, $p_2 = 73$, discrepancy: the last 100000 of Million data

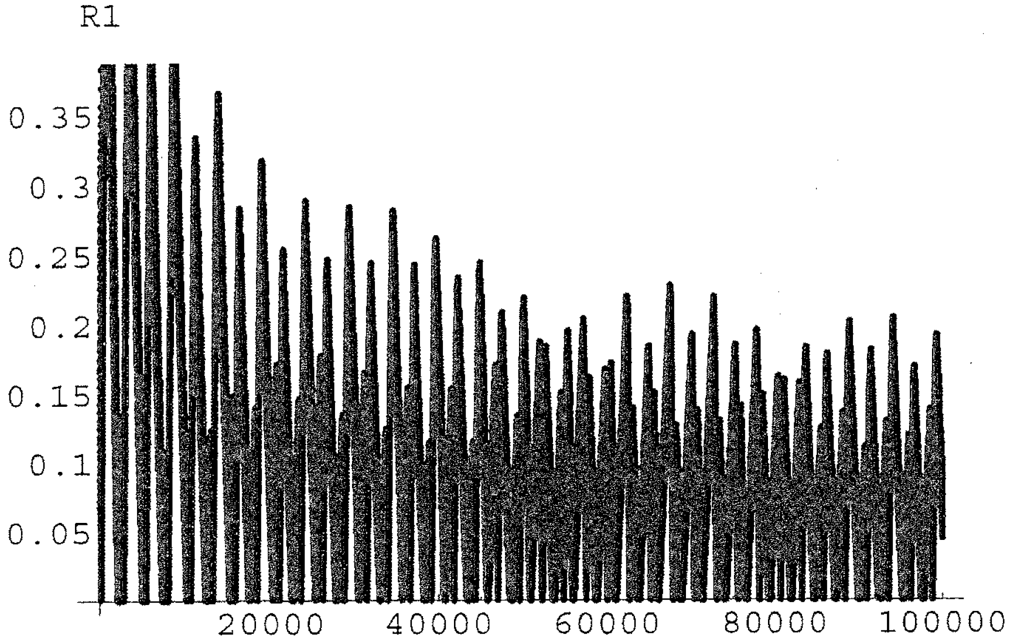

Figure 11: Halton $p_1 = 71$, $p_2 = 73$ ratio: the first 100000 of Million data

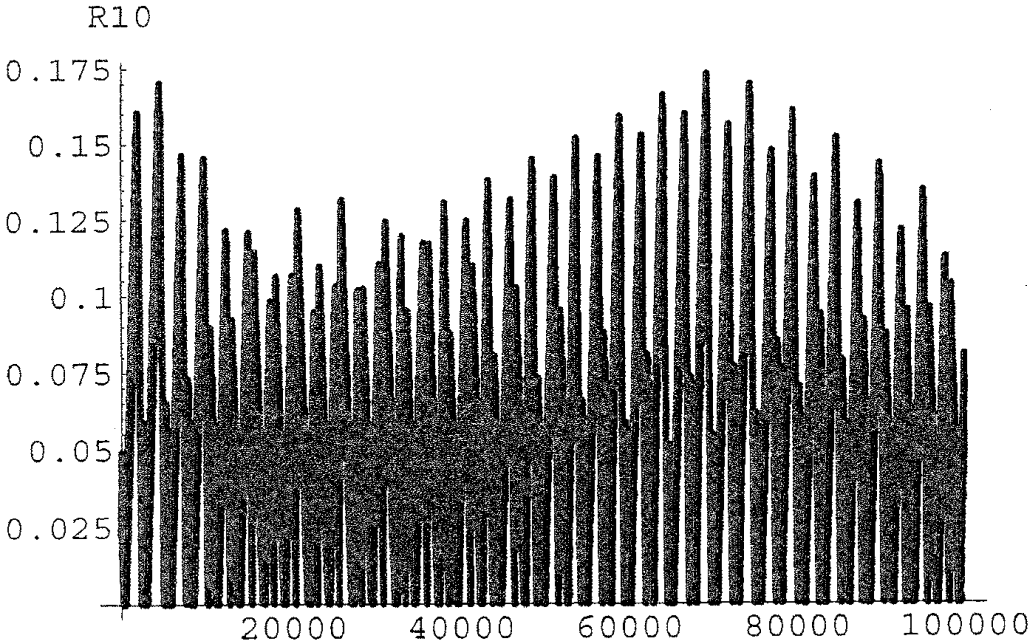

Figure 12: Halton $p_1 = 71$, $p_2 = 73$ ratio: the last 100000 of Million data

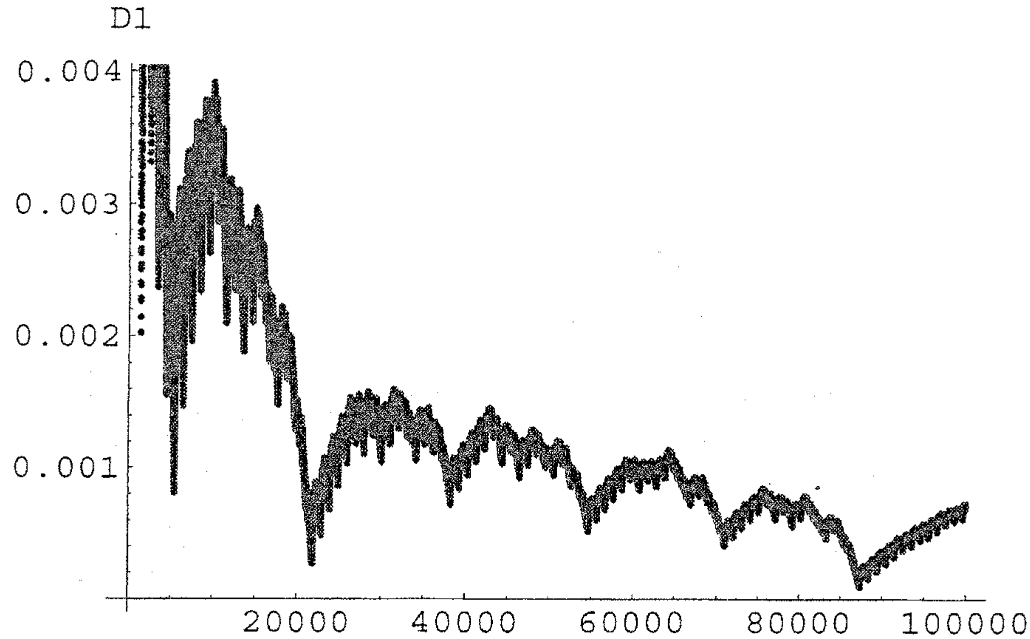

Figure 13: Mori, discrepancy: the first 100000

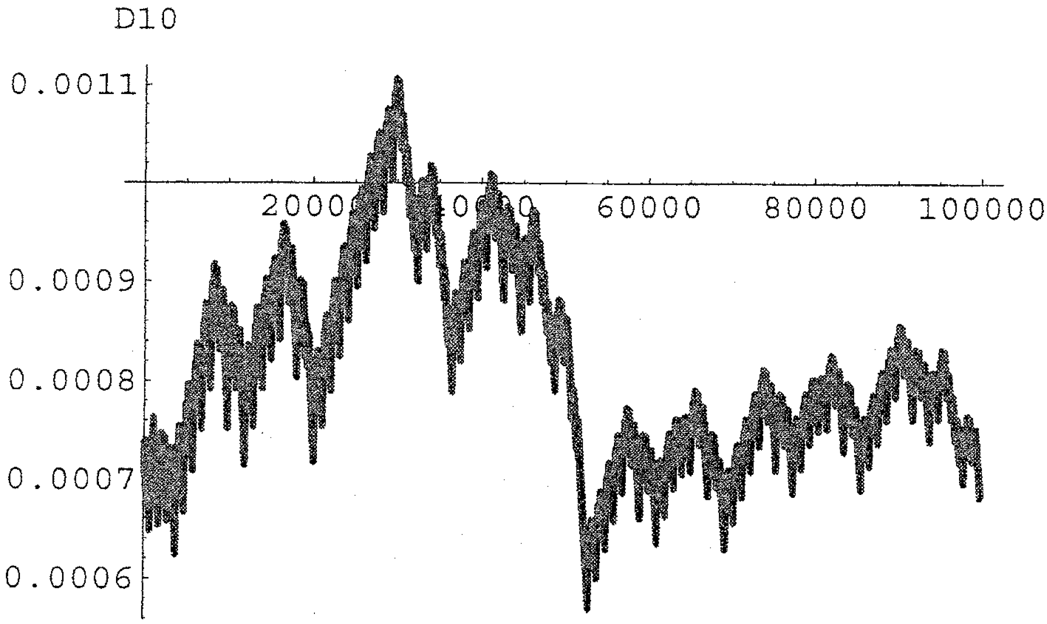

Figure 14: Mori, discrepancy: the last 100000

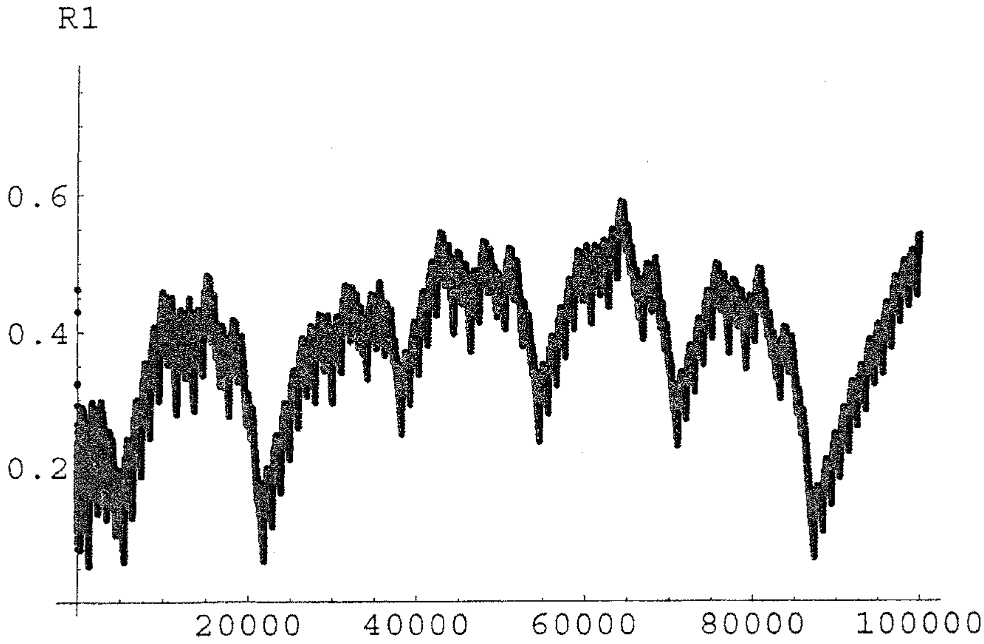

Figure 15: Mori , ratio: the first 100000

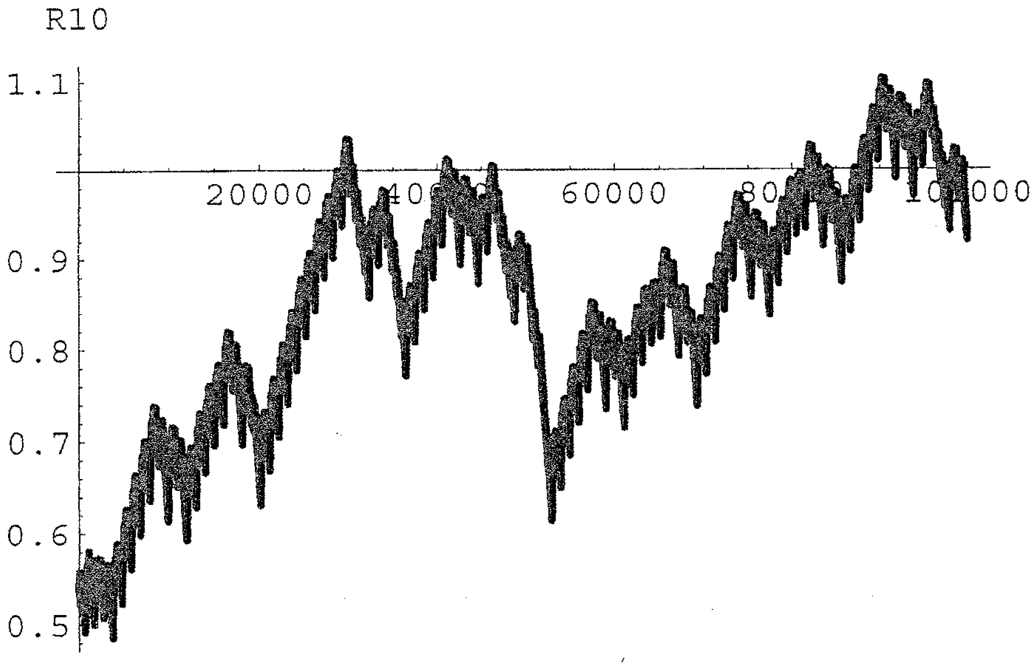

Figure 16: Mori, ratio: the last 100000

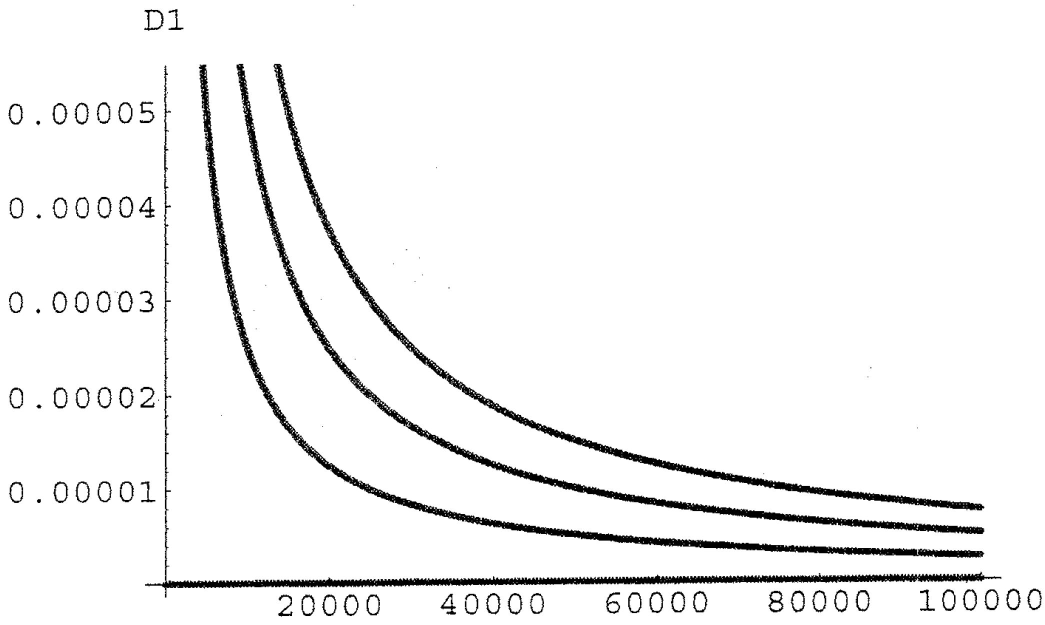

Figure 17: Mori (0.5, 0.5), discrepancy: the first 100000

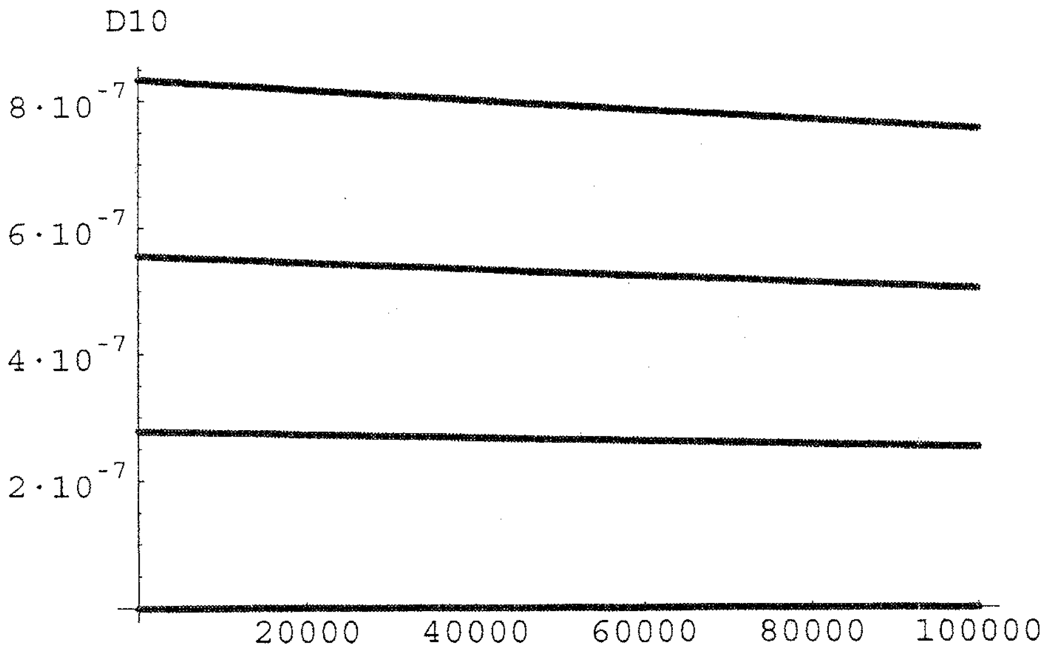

Figure 18: Mori (0.5, 0.5), discrepancy: the last 100000

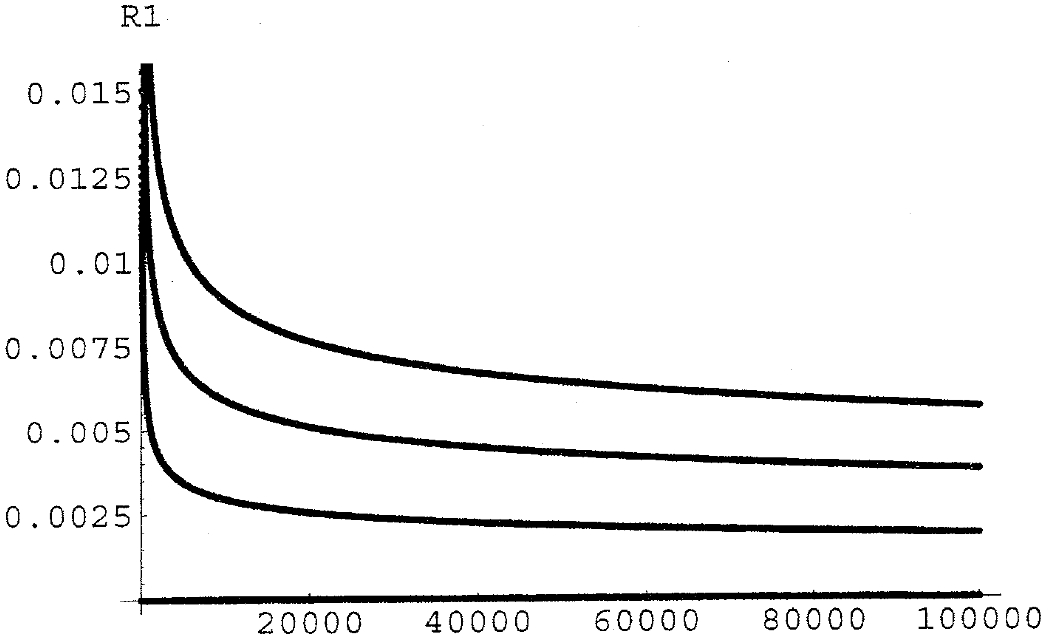

Figure 19: Mori (0.5, 0.5) ratio: the first 100000

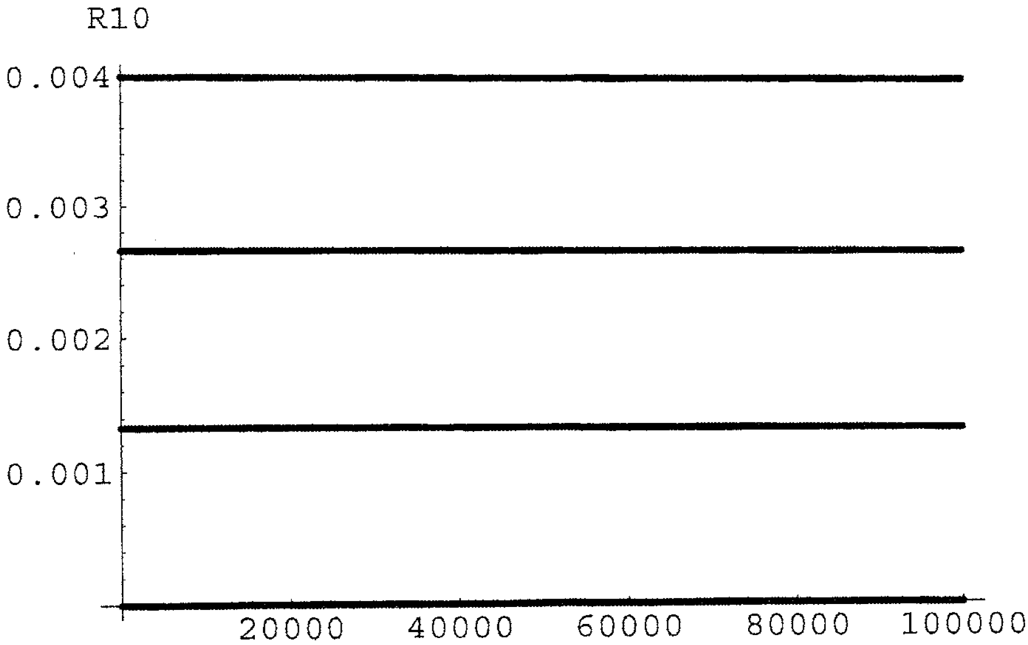

Figure 20: Mori (0.5, 0.5) ratio: the last 100000

References

[1] M.Mori, *Fredholm determinant for piecewise linear transformations*, Osaka J. Math., vol. **27**, 81-116 (1990).

[2] M.Mori, *Fredholm determinant for piecewise monotonic transformations*, Osaka J. Math., vol. **29**, 497-529 (1992).

[3] M.Mori, *Low discrepancy sequences generated by piecewise linear Maps*, Monte Carlo methods and Applications, vol.**4** No.2, 141-162 (1998).

[4] M.Mori, *Discrepancy of sequences generated by piecewise monotone Maps*, Monte Carlo Methods and Appl., vol. **5** No.1, 55-68(1999)

[5] M.Mori, *Construction of two dimensional low discrepancy sequences*, Monte Carlo methods and Applications vol. **8** No.2, 159-170 (2002)

[6] S.Ninomiya, *Constructing a new class of low-discrepancy sequences by using the β-adic transformations.* Math. Comput. Simul., vol. **47**, 405-420.(1998)

[7] S.Ninomiya, *On the discrepancy of the β-adic van der Corput sequence*, Journal of Mathematical Science, J. Math. Sci. Univ. Tokyo vol.**5**, 345-366(1998)